Calculus in the AP Physics C Course

The Derivative

Limits and Derivatives

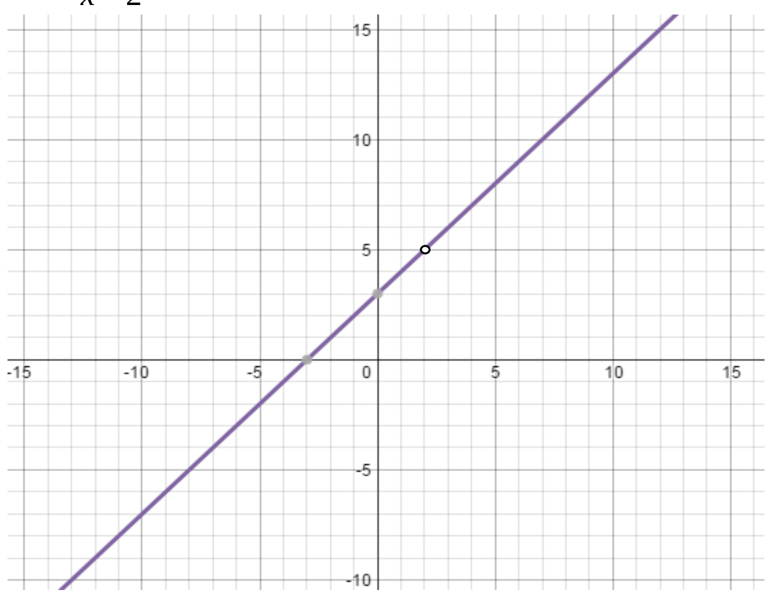
In physics, the ideas of the rate change of a quantity (along with the slope of a tangent line) and the area under a curve are essential. Limits are fundamental for the definitions of the two major concepts in calculus that describe these ideas: the derivative and the integral

Our intent here is to introduce these ideas to AP Physics C students who have most likely not yet covered them in a calculus course. Mathematical rigor has been left for the AP Calculus teacher to cover and a more intuitive approach is used here.

Limits

Limits are concerned with determining the values of functions based on their behavior near a value $x = a$. Often students are inclined to think that the value of a function determined from its behavior near $x = a$ is exactly the same as the value of the function at $x = a$. In teaching the notion of the limit we must make the distinction between behavior near $x = a$ and value at $x = a$.

Consider **Graph 1**: $f(x) = \dfrac{x^2 - 4}{x - 2} + 1$



If we begin taking x values to evaluate $f(x)$, we see that as x approaches 2, $f(x)$ approaches 5. Note that there is no $f(2)$, so we **can't** write $f(2) = 5$.

But as we get closer and closer to $x = 2$ (i.e., $x \to 2$), $f(x)$ gets closer and closer to the limit 5. We can write $\lim_{x \to 2} f(x) = 5$

Consider **Graph 2**: $f(x)=\dfrac{x-3}{2|x-3|}+4.5$

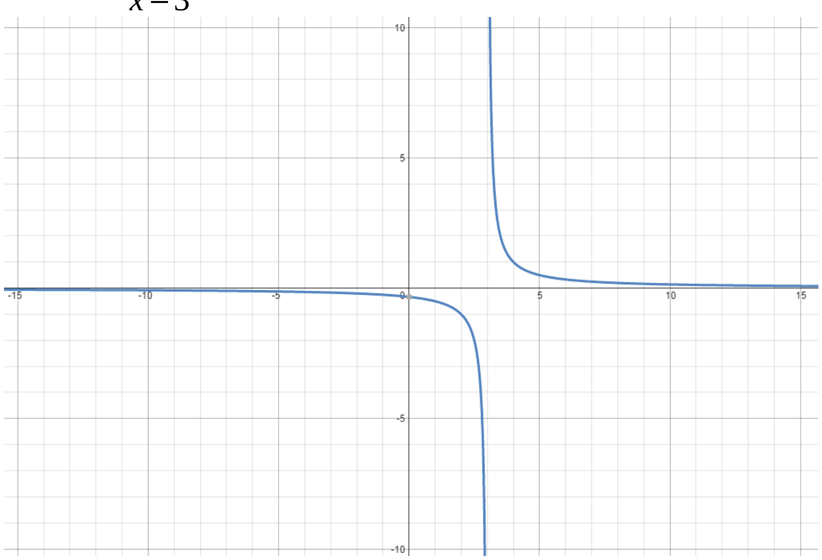
Note that there is no $f(3)$. This time we need to approach $x=3$ from both sides.

Approaching $x\to 3$ from the left (-): $\lim\limits_{x\to 3-} f(x)=4$

Approaching $x\to 3$ from the right (+): $\lim\limits_{x\to 3+} f(x)=5$

Again, we can get closer and closer to $x=3$, but we cannot compute $f(3)$ directly, and can only approach the value of $f(x)$ near $x=3$ by using the concept of the limit.

Infinite Limits: **Graph 3** $f(x)=\dfrac{1}{x-3}$



Note that the function is asymptotic to $x=3$, and there is no $f(3)$.

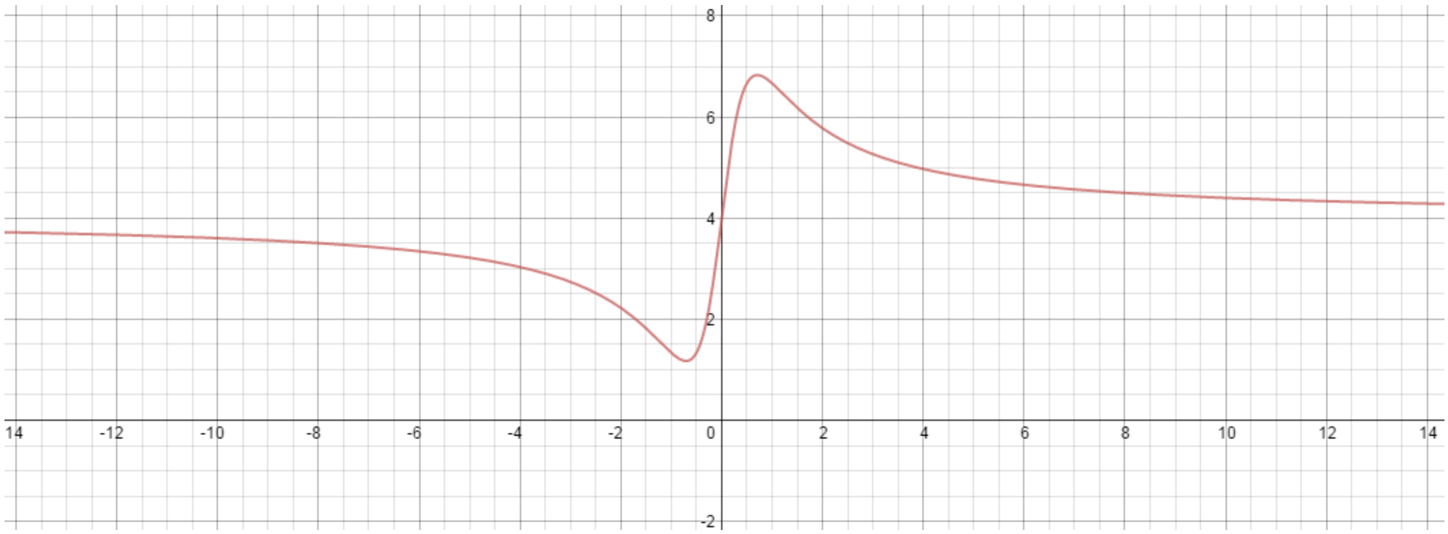
Approaching $x \to 3$ from the right (+): $\lim_{x \to 3+} f(x) = +\infty$ ($f(x)$ gets unboundedly large)

Approaching $x \to 3$ from the left (-): $\lim_{x \to 3-} f(x) = -\infty$ ($f(x)$ gets unboundedly large and negative)

There is no value for $\lim_{x \to 3} f(x)$ since the left and right limits don't agree.

We **do not** write $\lim_{x \to 3} f(x) = \pm\infty$.

Limits as $x \to \infty$: **Graph 4** $f(x) = \dfrac{4x}{x^2 + \frac{1}{2}} + 4$



Note that the limit of a function may be a particular value even if f never reaches that value. The limit must be approached, but not necessarily attained. We have $\lim_{x \to +\infty} f(x) = 4$, although $f(x)$ never attains 4.

Now for functions which are *continuous* at a particular point:

Theorem: If f is continuous at $x = a$, so that its graph does not break, then $\lim_{x \to a} f(x) = f(a)$

For example, recall Graph 2: $\lim_{x \to -1} f(x) = f(-1) = 4$

Another example: Consider the continuous function $f(x) = x^3 - 2x$

As $x \to 2$, $\lim f(x) = \lim(x^3 - 2x) = 2^3 - 2(2) = 4$.

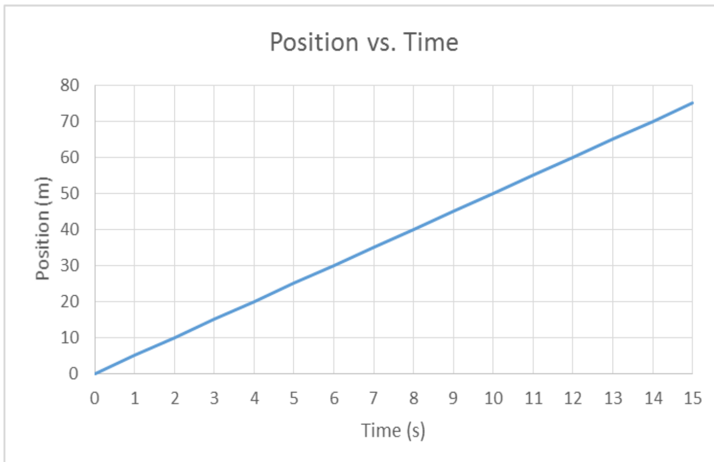
Which simply means $f(2) = 4$

Now, on to the **derivative**… get excited ☺

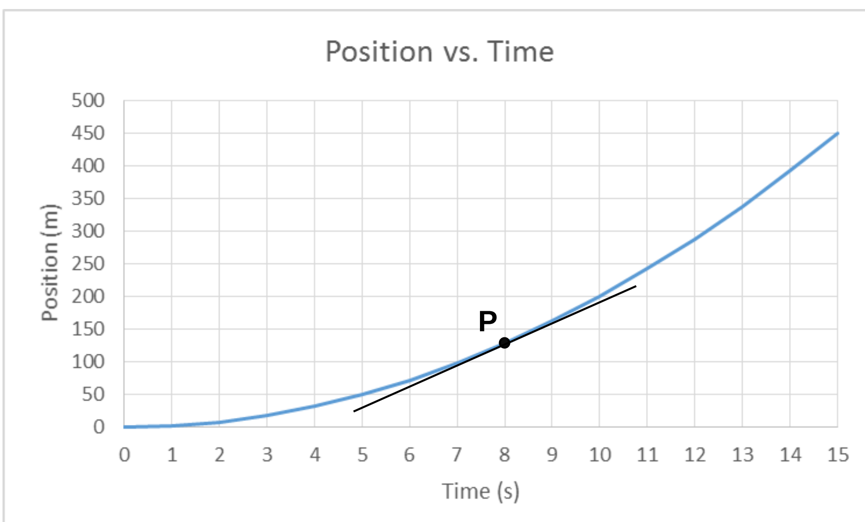
The Derivative as the Slope of a Tangent

The derivative of a function represents the rate of change of the function with respect to another quantity. The derivative also represents the slope of the line tangent to the graph of a function at a particular point.

Consider motion at a constant speed. The position, *x*, vs time, *t*, graph of this motion looks like this:



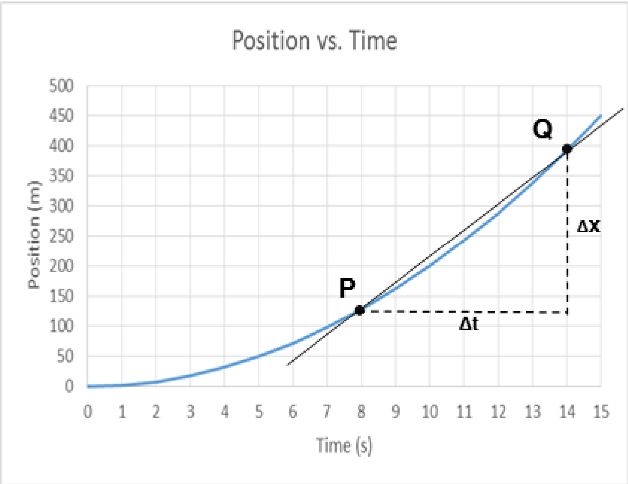
But what if the motion is accelerated? The speed (and thus slope) would be continually changing. We can approximate the speed at a particular point **P** by drawing a line tangent to the point and calculating its slope.

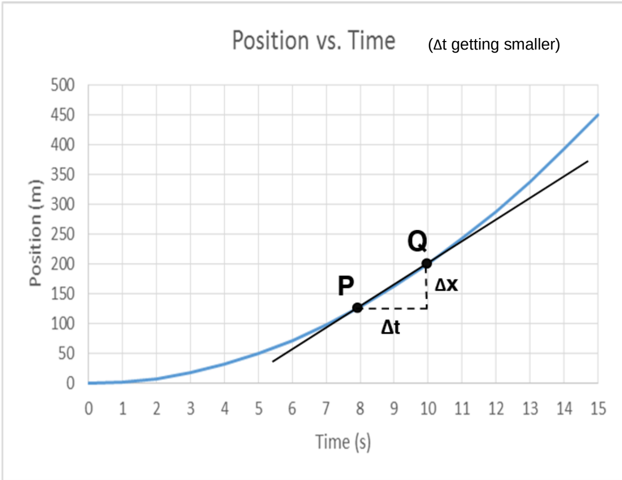


*slope of tangent line = instantaneous speed at point **P***

But how do we find the slope of a line tangent to a particular point on a curve? Choose a point **Q** on the curve near point **P** and connect the two points with a line called the secant line, as shown below. The slope of the secant line is the average speed between points **P** and **Q**, and is approximately equal to the instantaneous speed at point **P**.

If we allow point **Q** to approach point **P**, our $\Delta x's$ and $\Delta t's$ will become smaller and smaller (a limiting process as $\Delta t \to 0$) the secant line will become the tangent line at **P**, and its slope will represent the instantaneous speed at point **P**. The slope of a line tangent to a point **P** on a curve is called the *derivative* of the function describing the curve. We will return to this relationship later.





The Derivative as the *Rate of Change* of a Function

Suppose that the position of a car on a road at any time t is $x = f(t) = 12t - t^3$ so that at:

$t = 1$, $x = f(1) = 11$, at

$t = 2$, $x = f(2) = 16$, at

$t = 3$, $x = f(3) = 9$, and so on.

What is the speedometer reading v_{inst} at any time *t*? We know that

$$\text{average speed } v = \frac{\text{total change in position}}{\text{total change in time}}$$

Consider the period between times *t* and, $t + \Delta t$. The average velocity between these two times is

$$v = \frac{\text{change in position}}{\text{change in time}} = \frac{\text{later position} - \text{earlier position}}{\text{change in time}}$$

But how do we find the instantaneous velocity? Take smaller and smaller time intervals Δt, that is, allow Δt to approach zero and take the limit.

After expanding, we find that:

$$v_{inst} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{12(t+\Delta t) - (t+\Delta t)^3 - (12t - t^3)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{12t + 12\Delta t - (t^3 + 3t^2\Delta t + 3t\Delta t^2 + \Delta t^3) - 12t + t^3}{\Delta t} = \lim_{\Delta t \to 0} \frac{12\Delta t - 3t^2\Delta t - 3t\Delta t^2 - \Delta t^3}{\Delta t}$$

But, since we are taking the limit as $t \to 0$, we are left with

$$v_{inst} = \lim_{\Delta t \to 0}(12 - 3t^2 - 3t\Delta t - \Delta t^2) = 12 - 3t^2$$

The function, $12 - 3t^2$ is the derivative of *f* with respect to *t* and is denoted $f'(t)$.

$$x = f(t) = 12t - t^3$$

$$v_{inst} = f'(t) = 12 - 3t^2$$ (rate of change of *x* with respect to *t*)

(a) If the derivative (instantaneous velocity) is positive at a certain time, the car is moving to the right at that time.

(b) If the derivative is negative at a certain time the car is moving to the left at that time.

(c) What does it mean if the derivative at a certain time is zero?

(d) What is this car's speed at *t=0*?

We have seen that a derivative is the rate of change of a function and the slope of a line tangent to a point on a curve. Let's generalize the slope of a line tangent to a particular point on the curve. As done earlier we choose another point **Q** at the point [$x + \Delta x$, $f(x + \Delta x)$], and draw the secant line through **P** and **Q**. We now allow the secant line to become the tangent line at **P** using the limiting process as $\Delta x \to 0$.

As before, in terms of limits, derivative $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Then $f'(x) = \lim_{\Delta x \to 0} \dfrac{change\ in\ y}{change\ in\ x} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$

The derivative of *y* with respect to *x* can be written several different ways, but usually, we will write $\dfrac{dy}{dx}$ = derivative of *y* with respect to *x*. Keep in mind that $\dfrac{dy}{dx}$ is not a quotient (although we can sometimes treat it as one), but is only our notation for a derivative, our instantaneous rate of change.

Finding the derivative the easy way:

Recall from our previous example that

$x = f(t) = 12t - t^3$ and $v_{inst} = \dfrac{dx}{dt} = 12 - 3t^2$

In general, the derivative of $x^n = nx^{n-1}$

This is called the **POWER RULE** and it will get us pretty far in the application of differential calculus to physics.

Example 1.

$$y = 3x^4 + 5x^3 + 2x^2 + x + 6$$

Then, taking the derivative of *y* with respect to *x*, we write:

$$\frac{dy}{dx} = 12x^3 + 15x^2 + 4x + 1$$

Note that the derivative of a constant (in this case, 6) is always zero, since

(a) a constant by definition has no rate of change,
(b) graphically, the slope of a constant function is zero
(c) we could write *6* as $6x^0$, and its derivative would be $(0)(6x^{0-1}) = 0$

Example 2: Consider the position as a function of time: $x = 8t^3 + 3t^2 - 12$

(a) At $t = 0$, what is the position of the object?

This is pretty simple: we just substitute 0 in for t:

$$x = 8(0)^3 + 3(0)^2 - 12 = \boxed{-12}$$

(We really should put some units – most likely meters – in here, but I don't want to complicate this problem)

(b) At $t = 2\,s$, what is the velocity of the object?

This is a little more difficult, but we just demonstrated that the instantaneous velocity at any time t can be found by finding the derivative for the position vs. time function.

$$x = f(t) = 8t^3 + 3t^2 - 12$$

$$v_{inst} = f'(t) = 24t^2 - 6t$$

By substituting t = 2 seconds into the equation for v_{inst}, we get

$$v_{inst} = f'(2) = 24(2)^2 - 6(2) = \boxed{84}$$

(c) When is the velocity of the object zero?

If you think about what this question is asking, you'll see that solving this requires us to use the equation we have for the instantaneous velocity, v_{inst} equal to zero:

$$v_{inst} = f'(t) = 24t^2 - 6t = 0$$

The roots of this equation are $\boxed{t = 0\text{, and 4 seconds.}}$

(d) Is the object accelerating?

Simply looking at the equation for v_{inst}, we see that the velocity varies with time. Therefore, according to our definition for acceleration, the object *is* accelerating.

(e) Can we find its acceleration? We know that the slope of a velocity vs. time graph represents acceleration. That means, if we take the derivative of the velocity vs. time function, we will get an acceleration vs. time function. We sometimes call this the *second derivative* of the position vs. time function:

$$v_{inst} = f'(t) = 24t^2 + 6t$$

$$a_{inst} = f''(t) = 48t + 6$$

Note that the first derivative of the velocity function is also the second derivative of the position function. Using derivative notation,

<u>position:</u> $x = 8t^3 + 3t^2 - 12$

<u>velocity:</u> $v = \frac{dx}{dy} = 24t^2 + 6t$

<u>acceleration:</u> $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 48t + 6$

Alternate notation for the second derivative: $\mathrm{f}''(\mathrm{t}) = \frac{d^2x}{dt^2}$ does not indicate a ratio of squares, but only the notation for the second derivative.

Table of Derivatives

Most are Provided on AP Physics Equation Sheet

1. $\frac{d}{dx}(const.) = 0$

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(x^n) = nx^{n-1}$

4. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$

5. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

6. $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

7. $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$ (The Product Rule)

8. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$ (The Quotient Rule)

9. $\frac{d}{dx}(\sin x) = \cos x$

10. $\frac{d}{dx}(\cos x) = -\sin x$

11. $\frac{d}{dx}(\tan x) = \sec^2 x$

12. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

13. $\frac{d}{dx}e^x = e^x$

Derivative Practice

For problems 1-8 find the derivative of *y* with respect to *x* (i.e. $\frac{dy}{dx}$). Complete the assignment on a separate piece of graph paper.

1. $y = 5$

2. $y = x^4$

3. $y = 7x^8 + 9x^4 + 3x - 15$

4. $y = (2x^3 - 4x^2)(3x^5 + x^2)$

5. $y = \dfrac{2x^3 + 4}{x^2 - 4x + 1}$

6. $y = \dfrac{3}{x^5}$

7. $y = 3\sin x - 2\cos x$

8. $y = 5\cos 3x$

9. For the equation $y = x^2 - 4x - 3$ find
   (a) the equation of the slope of its tangent line at any point.
   (b) the equation of the tangent line at point (4,3) using point-slope form.

10. A particle undergoes straight-line motion with its displacement at any time given by the following equation (assume position in meters, and time in seconds) $y = 2t^3 - 4t^2 + 2t + 1$
    (a) Find the times when the particle is motionless.
    (b) Find the time when the particle is moving to the right.
    (c) Find the time when the particle is moving to the left.

11. The velocity of a particle moving along the x-axis for $t \geq 0$ is given by $v_x = 24 - 3t^3$.
    (a) What is the particle's acceleration when it first achieves a velocity of zero?
    (b) What is the particle's acceleration when it achieves its maximum displacement in the +x-direction?

12. The position of a particle moving along the x-axis is given by $x = 6t^2 - 2t + 4$.
    (a) What is the particle's velocity at times, $t = 2$ and $t = 4$?
    (b) What is the particle's average acceleration from $t = 2$ to $t = 4$